

# The Networked Common Goods Game

Jinsong Tan

Department of Computer and Information Science  
University of Pennsylvania, Philadelphia, PA 19104

## Abstract

We introduce a new class of games called the *networked common goods game* (NCGG), which generalizes the well-known *common goods game* [12]. We focus on a fairly general subclass of the game where each agent's utility functions are the same across all goods the agent is entitled to and satisfy certain natural properties (diminishing return and smoothness). We give a comprehensive set of technical results listed as follows.

- We show the optimization problem faced by a single agent can be solved efficiently in this subclass. The discrete version of the problem is however NP-hard but admits a *fully polynomial time approximation scheme* (FPTAS).
- We show uniqueness results of pure strategy Nash equilibrium of NCGG, and that the equilibrium is fully characterized by the structure of the network and independent of the choices and combinations of agent utility functions.
- We show NCGG is a *potential game*, and give an implementation of best/better response Nash dynamics that lead to fast convergence to an  $\epsilon$ -approximate pure strategy Nash equilibrium.
- Lastly, we show the *price of anarchy* of NCGG can be as large as  $\Omega(n^{1-\epsilon})$  (for any  $\epsilon > 0$ ), which means selfish behavior in NCGG can lead to extremely inefficient social outcomes.

## 1 Introduction

A collection of members belong to various communities. Each member belongs to one or more communities to which she can make contributions, either monetary or in terms of service but subject to a budget, and in turn benefits from contributions made by other members of the communities. The extent to which a member benefits from a community is a function of the collective contributions made by the members of this community.

A collection of collaborators are collaborating on various projects. Each collaborator is collaborating on one or more projects and each project has one or more collaborators. Each collaborator comes with certain endowment of resources, in terms of skills, time and energy, that she can allocate across the projects on which she is collaborating. The extent to which a project is successful is a function of the resources collectively allocated to it by its collaborators, and each of its collaborator in turn derives a utility from the successfulness of the project.

A collection of friends interact with each other, and friendships are reinforced through mutual interactions or weakened due to the lack of them. The more time and effort mutually devoted by two friends in their friendship, the stronger the friendship is; the stronger the friendship is, the more each benefit from it. However, each friend is constrained by her time and energy and has to decide how much to devote to each of her friends.

Suppose the community members, the collaborators and the friends (which we collectively call *agents*) are all self-interested and interested in allocating their limited resources in a way that maximizes their own total utility derived from the communities, projects, and mutual friendships

(which we collectively call *goods*) that they have access to. Interesting computational and economics questions abound: Can the agents efficiently find optimal ways to allocate their resources? Viewed as a game played by the agents over a bipartite network, how does the network structure affect the game? In particular, does there exist a pure strategy Nash equilibrium? Is it unique and will myopic and selfish behaviors of the agents lead to a pure strategy Nash equilibrium? And how costly are these myopic and selfish behaviors?

In this paper we address these questions by first proposing a model that naturally captures these strategic interactions, and then giving a comprehensive set of results to the scenario where there is only one resource to be allocated by the agents, and the utility an agent derives from a good to which she is entitled is a concave and smooth function of the total resource allocated to that good. We start by giving our model that we call the *networked common goods game* (NCGG).

**The Model.** The *networked common goods game* is played on a bipartite graph  $G = (P, A, E)$ , where  $P = \{p_1, p_2, \dots, p_n\}$  is a set of *goods* and  $A = \{a_1, a_2, \dots, a_m\}$  is a set of *agents*. If there is an edge  $(p_i, a_j) \in E$ , then agent  $a_j$  is entitled to good  $p_i$ . There is a single kind of divisible resource of which each agent is endowed with one unit (we note this is not a loss of generality as our results generalize easily to the case where different agents start with different amounts of resource). Moreover, we can assume Nature has endowed each common good  $p_i$  with  $\alpha_i$  amount of resource that we call the *ground level*; this can be viewed as modelling  $p_i$  as having access to some external sources of contributions.

Let  $\mathcal{N}(v)$  denote the set of neighbors of a node  $v \in P \cup A$ ,  $x_{ij} \in [0, 1]$  the amount of resource agent  $a_j$  contribute to good  $p_i$  and  $\omega_i = \alpha_i + \sum_{a_k \in \mathcal{N}(p_i)} x_{ik}$  the total amount of resource allocated to good  $p_i$ . Each agent  $a_j$  derives certain utility  $\mathcal{U}_j(\omega_i)$  from each  $p_i$  of which she is a member. We always assume  $\mathcal{U}_j(0) = 0$  and for the most part of the paper, we consider the case where  $\mathcal{U}_j(\cdot)$  is increasing, concave and differentiable. Being self-interested, agent  $a_j$  is interested in allocating her resources across the goods to which she is entitled in a way that maximizes her total utility  $\sum_{p_i \in \mathcal{N}(a_j)} \mathcal{U}_j(\omega_i)$ .

**Our Results.** We first consider the optimization problem faced by a single agent: Given the resources already allocated to the goods to which agent  $a_j$  is entitled, find a way to allocate resource so that  $a_j$ 's total utility is maximized. We call this the *common goods problem* (CGP) and consider both continuous and discrete versions, where the agent's resource is either infinitely divisible or atomic.

- We show that for the continuous version, if  $\mathcal{U}_j(\cdot)$  is assumed to be increasing, concave and differentiable, then CGP has an analytical solution. On the other hand, the discrete version of CGP is NP-hard but admits an FPTAS.

We then turn to investigate the existence and uniqueness of pure strategy Nash equilibrium of NCGG<sup>1</sup>. We consider two concepts of uniqueness of equilibrium, among which *strong uniqueness* is the standard concept of equilibrium uniqueness whereas *weak uniqueness* is defined as follows: For any two equilibria  $\mathcal{E}$  and  $\mathcal{E}'$  of the game and for any good  $p_i \in P$ , the total amount of resource allocated to  $p_i$  is the same under both  $\mathcal{E}$  and  $\mathcal{E}'$ . We have the following results.

- We show for any NCGG instance, a Nash equilibrium always exists. And we show that this Nash equilibrium is weakly unique not only in a particular NCGG instance, but across all NCGG instances played on the same network as long as the utility function of each agent is increasing, concave and differentiable. And if in addition the underlying graph is a tree,

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<sup>1</sup>Since we concern ourself with only pure strategy Nash equilibrium in this paper, we use it interchangeably with Nash equilibrium.

the equilibrium is strongly unique. Our results do not assume that different agents have the same utility function; this demonstrates that Nash equilibrium in NCGG is completely characterized by network structure.

We also consider the convergence of Nash dynamics of the game, and its *price of anarchy*: The worst-case ratio between the social welfare of an optimal allocation of resources and that of a Nash equilibrium [17].

- We show that NCGG is a *potential game*, a concept introduced in [16], therefore any (better/best response) Nash dynamics always converge to the (unique) pure strategy Nash equilibrium. We then propose a particular implementation of Nash dynamics that leads to fast convergence to a state that is an additive  $\epsilon$ -approximation of the pure strategy Nash equilibrium of NCGG. The convergence takes  $O(Kmn)$  time, where  $K = \max_j \mathcal{U}_j^{-1}(\epsilon/n)$ , which for most reasonable choices of  $\mathcal{U}_j$  is a polynomial of  $n$  and  $m$ . (For example, for  $\mathcal{U}_j(x) = x^p$  where  $p \in (0, 1)$  is a constant, it is sufficient to set  $K = (n/\epsilon)^{1/p}$ , which is a polynomial in  $n$ .)
- We show the price of anarchy of the game is  $\Omega(n^{1-\epsilon})$  (for any  $\epsilon > 0$ ), which means selfish behavior in this game can lead to extremely inefficient social outcomes, for a reason that echoes the phenomenon of *tragedy of the commons* [11].

We note that NCGG introduced in this paper has the particularly nice property that very little is assumed about agents' utility functions. Unlike most economic models considered in the literature where not only a particular form of utility function is assumed about a particular agent, but very often the same utility function is imposed across all agents, so that the model remains mathematically tractable, our model do not assume more than the following: 1)  $\mathcal{U}_j(0) = 0$ ; 2)  $\mathcal{U}_j(\cdot)$  has diminishing return (increasing and convex); 3)  $\mathcal{U}_j(\cdot)$  is smooth (differentiable). In particular, we do not need to assume different agents share a common utility function for our results to go through.

**Related Work.** The networked common goods game we consider is a natural generalization of the well-known common goods game [12]. Bramoullé and Kranton considered a different generalization of the common goods game to networks [3]. In their formulation a (general, non-bipartite) network is given where each node represents an agent  $a_i$ , who can exert certain amount of effort  $e_i \in [0, +\infty)$  towards certain common good and such effort incurs a cost of  $ce_i$  on the part of the agent, for some constant  $c$ .  $a_i$ 's effort directly benefits another agent  $a_j$  iff they are directed connected in the network, and the utility of  $a_i$  is defined as  $\mathcal{U}_i(e_i + \sum_{a_j \in \mathcal{N}(i)} e_j) - ce_i$ . Bramoullé and Kranton then analyze this model to yield the following interesting insights: First, in every network there is an equilibrium where some individuals contribute whereas others free ride. Second, specialization can be socially beneficial. And lastly, a new link in the network can reduce social welfare as it can provide opportunities to free ride and thus reduce individual incentives to contribute. We note both the model and the research perspectives are very different from those considered in this paper.

A more closely related model is that studied by Fol'gardt [8, 9]. The author considered a resource allocation game played on a bipartite graph that is similar to our setting. In Fol'gardt's model, each agent has certain amount of discrete resources, each of unit volume, that she can allocate across the 'sites' that she has access to. Each site generates certain utility for the agent, depending on the resources jointly allocated to it by all its adjacent agents. In Fol'gardt's formulation, each agent is interested in maximizing the minimum utility obtained from a single site she has access to. The analysis of Fol'gardt's resource allocation game is limited to very specific and small graphs [8, 9].

A variety of other models proposed and studied in the literature bear similarities to the networked common goods game considered here. These include Fisher's model of economy [7], the

bipartite exchange economy [13, 4], the fixed budget resource allocation game [6, 18], the Pari-Mutuel betting as a method of aggregating subjective probabilities [5], and the market share game [10]. However these model all differ significantly in the ways allocations yield utility.

## 2 The Common Goods Problem

Recall that CGP is the optimization problem faced by a single agent: An agent has access to  $n$  goods, each good  $p_i$  has already been allocated  $\alpha_i \geq 0$  resources. The agent has certain amount of resource to allocate across the  $n$  goods. Denote by  $x_i$  ( $i = 1, \dots, n$ ) the amount of resource the agent allocates to goods  $i$ , she receives a total utility of  $\sum_{i=1}^n \mathcal{U}(\alpha_i + x_i)$ . In this section, we consider two versions of this optimization problem, where the resource is either infinitely divisible or discrete.

### 2.1 Infinitely Divisible Resource

Without loss of generality, assume the agent has access to one unit of resource. In the infinitely divisible case, CGP is a convex optimization problem captured by the following convex program.

$$\begin{aligned} & \text{maximize} && \sum_{i=1}^n \mathcal{U}(\alpha_i + x_i) \\ & \text{subject to} && \sum_{i=1}^n x_i = 1 \\ & && x_i \geq 0 \quad (i = 1, 2, \dots, n) \end{aligned} \tag{1}$$

where the constraint  $\sum_{i=1}^n x_i = 1$  comes from the observation that  $\mathcal{U}(\cdot)$  is an increasing function so an optimal solution must have allocated the entire unit of resource.

As it turns out, as long as  $\mathcal{U}(\cdot)$  is increasing, concave and differentiable, the above convex program admits exactly the same unique solution regardless of the particular choice of  $\mathcal{U}(\cdot)$ . And we note this solution coincides with what is known in the literature as the *water-filling* algorithm [2]. This is summarized in the following theorem. The proof relies on the above program being convex to apply the well-known *Karush-Kuhn-Tucker* (KKT) optimality condition [2], and is relegated to the appendix.

**Theorem 1** *For any utility function  $\mathcal{U}$  that is concave and differentiable, the convex program admits a unique analytical solution. Moreover, the solution is unique across all choices of  $\mathcal{U}(\cdot)$  as long as it is increasing, concave and differentiable.*

Therefore the unique optimal way to allocate resources across the goods is independent of the agent's utility function as long as it is differentiable and has diminishing return, which is a very reasonable assumption. We note this is a particularly nice property of the model as it frees us from imposing any particular form of utility function, which can often be arbitrary, and the risk of observing artifacts thus introduced. In NCGG considered later, this property frees us from making the assumption that each agent has the same utility function, which is standard of most economic models whose absence would often render the underlying model intractable.

### 2.2 Discrete Resource

In the discrete case, the agent has access to a set of atomic resources, each of integral volume. We show in the next two theorems that although the discrete CGP is NP-hard even in a rather special case, the general problem always admits an FPTAS.

**Theorem 2** *The discrete common goods problem is NP-hard even when each atomic resource is of unit volume and  $\mathcal{U}(\cdot)$  is increasing.*

**Proof:** We prove the hardness result by giving a reduction from the NP-hard *unbounded knapsack problem* [15].

UNBOUNDED KNAPSACK PROBLEM (UKP)

INSTANCE: A finite set  $U = \{1, 2, \dots, n\}$  of items, each item  $i$  has value  $v_i \in \mathbb{Z}^+$ , weight  $w_i \in \mathbb{Z}^+$  and unbounded supply, a positive integer  $B \in \mathbb{Z}^+$ .

QUESTION: Find a multi-subset  $U'$  of  $U$  such that  $\sum_{i \in U'} v_i$  is maximized and  $\sum_{i \in U'} w_i \leq B$ .

Since supply is unlimited we can assume without loss of generality that no two items are of the same weight and no item is strictly dominated by any other item, i.e.  $w_i > w_j$  implies  $v_i > v_j$ . Now create  $n$  goods,  $p_1, \dots, p_n$ , where  $p_i$  corresponds to item  $i$  and has a ground level  $(i-1)B$ . Let the agent have access to a total of  $B$  atomic resource, each of unit volume. Define the utility function  $\mathcal{U}(\cdot)$  as follows:  $\mathcal{U}(\omega) = \sum_{i=1}^{\mu(\omega)-1} \lfloor \frac{B}{w_i} \rfloor v_i + \lfloor \frac{\nu(\omega)}{w_{\mu(\omega)}} \rfloor v_{\mu(\omega)} + \frac{\omega}{(n^2-n+2)B^2}$  where  $\mu(\omega) = \lceil \omega/B \rceil$  and  $\nu(\omega) = \omega \bmod B$ .

Clearly,  $\mathcal{U}(\cdot)$  is a strictly increasing function, and thus we only concern ourselves with those CGP solutions that allocate all  $B$  atomic units of resources. One can then verify that there is a solution of total value  $K$  to the UKP instance iff there is a solution of total utility  $\sum_{j=1}^n \sum_{i=1}^{j-1} \lfloor \frac{B}{w_i} \rfloor v_i + \frac{1}{2B} + K$  to the corresponding CGP instance. Therefore the discrete common goods problem is NP-hard. ■

**Theorem 3** *The discrete common goods problem always admits an FPTAS.*

**Proof:** The discrete common goods problem can be reduced to the *multiple-choice knapsack problem*

MULTIPLE-CHOICE KNAPSACK PROBLEM (MCKP)

INSTANCE: A finite set  $U = \{1, 2, \dots, k\}$  of items, each item  $i$  has value  $v_i$ , weight  $w_i$  and belongs to one of  $n$  classes, a capacity  $B > 0$ .

QUESTION: Find a subset  $U'$  of  $U$  such that  $\sum_{i \in U'} v_i$  is maximized,  $\sum_{i \in U'} w_i \leq B$ , and at most one item is chosen from each of the  $n$  classes.

The reduction goes as follows. For a general CGP instance, where there are  $B$  atomic unit-volume resources, and goods  $\{p_1, \dots, p_n\}$  such that good  $p_i$  has ground level  $\alpha_i$ , create a MCKP instance such that there are  $n$  classes  $c_1, \dots, c_n$ . Class  $c_i$  corresponds to good  $p_i$  and has  $B$  items of weight  $j$  and value  $\mathcal{U}(\alpha_i + j)$ , for  $j = 1, \dots, B$ . The knapsack is of total capacity  $B$ .

It is not hard to see that there is a solution of total utility  $K$  to the CGP instance if and only if there is a solution of total value  $K$  to the MCKP instance. Therefore, any approximation algorithm for the latter translates into one for the former with the same approximation guarantee. Since an FPTAS is known for MCKP [1, 14], CGP also admits an FPTAS. ■

### 3 Pure Strategy Nash Equilibrium

We consider in this section the existence and uniqueness of Nash equilibrium in NCGG.

#### 3.1 The Existence of Nash Equilibrium

First we show a Nash equilibrium always exists in NCGG when the utility functions satisfy certain niceness properties.

**Theorem 4** *For any NCGG instance, a pure strategy Nash equilibrium always exists as long as  $\mathcal{U}_j$  is increasing, concave and differentiable for any agent  $a_j$ .*

**Proof:** Let  $\deg(a_i)$  be the degree of agent  $a_i$  and  $D = \sum_{a_i \in A} \deg(a_i)$ . Let  $s \in [0, 1]^D$  be the state vector that corresponds to how the  $m$  agents have allocated their resources, where the  $(\sum_{k=1}^{i-1} \deg(a_k))$ th to the  $(\sum_{k=1}^i \deg(a_k))$ th dimension of  $s$  correspond to  $a_i$ 's allocation of her resource on the  $\deg(a_i)$  goods she is connected to (assume an arbitrary but fixed order of the goods  $a_i$  is connected to). Define function  $f : [0, 1]^D \rightarrow [0, 1]^D$  such that  $f(s)$  maps to the *best response* state  $s'$ , where  $(s'_{\sum_{k=1}^{i-1} \deg(a_k)}, \dots, s'_{\sum_{k=1}^i \deg(a_k)})$  corresponds to  $a_i$ 's best response. Note  $s'$  is unique because each agent  $a_i$ 's best response is unique by Theorem 1, therefore  $f(s)$  is well-defined.

It is clear that  $[0, 1]^D$  is compact (i.e. closed and bounded) and convex, and  $f$  is continuous. Therefore, applying Brouwer's fixed point theorem shows that  $f$  has a fixed point, which implies NCGG has a Nash equilibrium. ■

We note on the other hand, it is easy to see that if  $\mathcal{U}_j$  is allowed to be convex, then a pure strategy Nash equilibrium may not exist in NCGG.

### 3.2 The Uniqueness of Nash Equilibrium

We next establish uniqueness results of Nash equilibrium of NCGG in the next two theorems. Apparently, NCGG played on a general graph does not have a unique Nash equilibrium in the standard sense: Consider for example the  $2 \times 2$  complete bipartite graph where  $P = \{p_1, p_2\}$  and  $A = \{a_1, a_2\}$ , for any  $0 \leq \delta \leq 1$ ,  $a_1$  (resp.  $a_2$ ) allocating  $\delta$  (resp.  $1 - \delta$ ) resource on  $p_1$  and  $1 - \delta$  (resp.  $\delta$ ) resource on  $p_2$  constitutes a pure strategy Nash equilibrium and therefore there are uncountably infinite many of them. However, all these equilibria can still be considered as equivalent to each other in the sense that they all allocate exactly the same amount of resource to each good. And the reader is encouraged to verify as an exercise that any Nash equilibrium in the above NCGG instance belongs to this equivalence class. Therefore, the Nash equilibrium is still unique, albeit in a weaker sense.

To capture this, we thus consider two concepts of uniqueness of equilibrium: We say an NCGG instance has a *weakly unique* equilibrium if all its equilibria allocate exactly the same amount of resource on each good  $p_i$ . And if an NCGG instance has an equilibrium that is unique in the standard sense, we call it *strongly unique*. We note the concept of weak uniqueness is a useful one as it implies the uniqueness of each agent's utility in equilibrium, which is really what we ultimately care about.

We show two uniqueness results in this section. The first one establishes that NCGG has a strongly unique Nash equilibrium if the underlying graph is a tree. The second one indicates that it is not a coincidence that the example shown above has a weakly unique equilibrium — in fact, we show *any* NCGG instance has a weakly unique Nash equilibrium. Furthermore, our results indicate that the equilibrium is a function of the structure of the underlying graph only, and independent of the particular forms and combinations of agents' utility functions, as long as these functions are increasing, concave and differentiable.

**Theorem 5** *The Nash equilibrium of NCGG is weakly unique across all networked common goods games played on a given bipartite graph  $G = (P, A, E)$ , as long as  $\mathcal{U}_j$  is increasing, convex and differentiable for any agent  $a_j$ .*

**Proof:** Suppose otherwise that there are two equilibria  $\mathcal{E}$  and  $\mathcal{E}'$  that have different amount of resource  $\omega_i$  and  $\omega'_i$  allocated to some good  $p_i$  (throughout the rest of the paper whenever it is clear from the context, for any good  $p_x$  we denote by  $\omega_x$  and  $\omega'_x$  the amount of resource allocated to  $p_x$  in  $\mathcal{E}$  and  $\mathcal{E}'$ , respectively). Without loss of generality assume  $\omega'_i < \omega_i$ . Then there must exists some agent  $a_j \in \mathcal{N}(p_i)$  who is allocating less resource on  $p_i$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ , and as a result,  $a_j$  must be allocating more resource on some good  $p_k \in \mathcal{N}(a_j) \setminus \{p_i\}$  in  $\mathcal{E}'$  because in equilibrium each

agent allocates all of its resources. The fact that  $a_j$  is allocating nonzero resource on  $p_i$  in  $\mathcal{E}$  implies  $\omega_i \leq \omega_k$ , and for the same reason  $\omega'_k \leq \omega'_i$ . Therefore we have  $\omega_k - \omega'_k \geq \omega_i - \omega'_i > 0$ .

Now consider the following process: Starting from set  $S_0 = \{p_i\}$ , add goods to  $S_0$  that share an agent with  $p_i$  and whose total resource have decreased by at least  $\omega_i - \omega'_i$  in  $\mathcal{E}'$ ; let the new set be  $S_1$ . Then grow the set further by adding goods that share an agent with some good in  $S_1$  and whose total resource are reduced by at least  $\omega_i - \omega'_i$  in  $\mathcal{E}'$ . Continue this process until no more goods can be added and let the resulting set be  $S$ . By construction every good in  $S$  has its total resource decreased by at least  $\omega_i - \omega'_i$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ ; in fact, it can be shown that the decrease is exactly  $\omega_i - \omega'_i$  for each good in  $S$ .

If  $S = P$ , then we have a contradiction immediately because if each good in  $P$  has its total resource decreased by a positive amount in  $\mathcal{E}'$  then it implies the agents collectively have a positive amount of resources not allocated, contradicting the fact that  $\mathcal{E}'$  is a Nash equilibrium.

We now claim that indeed  $S = P$ . Suppose otherwise  $P = S \cup T$  and  $T \neq \emptyset$ . Then,  $\mathcal{N}(S)$ , the neighboring agents of  $S$  are collectively spending less resources on  $S$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ , which implies there exists an agent  $a \in \mathcal{N}(S)$  who is allocating more resources to a good  $p_t \in T$  in  $\mathcal{E}'$  than in  $\mathcal{E}$  and less resources to a good  $p_s \in S$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ . By an argument similar to one given above, we have  $\omega_s \leq \omega_t$  and  $\omega'_t \leq \omega'_s$ , and thus  $\omega_t - \omega'_t \geq \omega_s - \omega'_s \geq \omega_i - \omega'_i$ . This implies that  $p_t$  should be in  $S$  rather than  $T$ ; so we must have  $T = \emptyset$  or  $S = P$ .

Therefore  $\mathcal{E}$  and  $\mathcal{E}'$  must be equivalent in the sense that for any good  $p_i \in P$ ,  $\omega_i = \omega'_i$ ; this allows us to conclude that the Nash equilibrium of NCGG on any graph is *weakly unique*. ■

Next, we move to establish the strong uniqueness result on trees. We need the following lemma before we proceed to the main theorem of the section.

**Lemma 1** *For any instance of NCGG on a tree  $G = (P, A, E)$ , let  $\mathcal{E}$  be a Nash equilibrium of this game,  $\alpha_i$  the ground level of  $p_i \in P$  and  $\omega_i$  the total resource allocated on  $p_i$  in  $\mathcal{E}$ . For any other instance of NCGG where everything is the same except that  $\alpha_i$  is increased, if  $\mathcal{E}'$  is an equilibrium of this new instance and  $\omega'_i$  is total resource allocated to  $p_i$  in  $\mathcal{E}'$ , then  $\omega'_i \geq \omega_i$ .*

**Proof:** Without loss of generality assume all leafs of the tree are goods (because a leaf agent has no choice but to allocate all her resources to the unique good she is connected to) and root the tree at  $p_i$ . Suppose  $\omega'_i < \omega_i$ . Since  $\alpha'_i > \alpha_i$ , it must be the case that there exists some agent  $a_j \in \mathcal{N}(p_i)$  who is allocating less resource on  $p_i$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ , this in turn implies that  $a_j$  is allocating more resource to some good  $p_k \in \mathcal{N} \setminus \{p_i\}$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ . Therefore we have  $\omega_i \leq \omega_k$  and  $\omega'_k \leq \omega'_i$  and thus  $\omega_k - \omega'_k \geq \omega_i - \omega'_i > 0$ . If  $k$  is a leaf then this is obviously a contradiction. Otherwise, we can continue the above reasoning recursively and eventually we will reach a contradiction by having a leaf good whose total resource decreases in  $\mathcal{E}'$  whereas at the same time its unique neighboring agent is allocating more resources to it. ■

**Theorem 6** *The Nash equilibrium is strongly unique across all NCGG played on a given tree  $G = (P, A, E)$ , as long as  $\mathcal{U}_j$  is increasing, convex and differentiable for any agent  $a_j$ .*

**Proof:** Again without loss of generality assume leafs are all goods. We have the following claim.

Claim. *For any NCGG instance on a tree  $G = (A, P, E)$ , if there is an equilibrium  $\mathcal{E}$  where total resource allocated is the same across all goods, then  $\mathcal{E}$  is the strongly unique Nash equilibrium.*

Proof. Suppose  $\mathcal{E}$  is not strongly unique. Let  $\mathcal{E}'$  be a different Nash equilibrium. By Theorem 5  $\mathcal{E}'$  can only be weakly different from  $\mathcal{E}$ . Since  $\mathcal{E}$  and  $\mathcal{E}'$  are weakly different there must exist edge  $(p_i, a_j)$  such that  $a_j$  is allocating different amount of resource in  $\mathcal{E}$  and  $\mathcal{E}'$ ; without loss of generality, assume  $a_j$  is allocating less resource in  $\mathcal{E}'$  than in  $\mathcal{E}$ . Root the tree at  $p_i$ , then  $a_j$  must be allocating more resource in  $\mathcal{E}'$  to one of its child  $p_k \in \mathcal{N}(a_j) \setminus \{p_i\}$ . Note given the amount of resource allocated by  $a_j$  on  $p_k$ , the game played at the subtree rooted at  $p_k$  can be viewed as

independent of the game played in the rest of the tree, by viewing the resource allocated by  $a_j$  on  $p_k$  as part of the ground level of  $p_k$ . Now that the ground level has increased, by Lemma 1 any equilibrium on the subtree rooted at  $p_k$  must not have the total resource allocated on  $p_k$  decreased, so we have  $\omega'_k \geq \omega_k$ . If  $\omega'_k > \omega_k$ , then this is a contradiction to weak uniqueness. If  $\omega'_k = \omega_k$ , then one of  $p_k$ 's child must be allocating less resource to  $p_k$  in  $\mathcal{E}'$  than in  $\mathcal{E}$  and we can repeat the above reasoning recursively. Continue this process until we either reach the conclusion that  $\mathcal{E}$  and  $\mathcal{E}'$  are strongly different, which is a contradiction, or reach a leaf good whose allocated resource in  $\mathcal{E}'$  is the same as that in  $\mathcal{E}$  even when his unique neighboring agent is allocating more resource to it in  $\mathcal{E}'$ , which is again a contradiction.

Resume Proof of Theorem. We prove this theorem by giving an induction on the size of the tree  $N = |A| + |P|$ . First note the equilibrium is unique when  $N \leq 2$  (in the trivial case where either  $E = \emptyset$ , the claim is vacuously true). Assume the theorem is true for any tree of size  $N \leq K$ , consider the case  $N = K + 1$ .

For any instance  $G_{K+1}$  with  $N = K + 1$ , let  $\mathcal{E}$  be a Nash equilibrium (whose existence is implied by Theorem 4). We want to show that  $\mathcal{E}$  is strongly unique. Let

$$E(\mathcal{E}) = \{(p_i, a_j) \mid \omega_i > \omega_k \text{ and } x_{jk} > 0 \text{ in } \mathcal{E}\}$$

If  $E(\mathcal{E}) = \emptyset$  then it must be the case that the total resource allocated is the same across all goods, and by the above claim  $\mathcal{E}$  is thus strongly unique and we are through. Otherwise, partition  $G$  into subtrees by removing  $E(\mathcal{E})$  from  $E$ . Note the size of each subtree thus resulted is at most  $N$ , so by induction they each has a strongly unique equilibrium; this implies that if we can prove  $E(\mathcal{E}') = E(\mathcal{E})$  for any equilibrium  $\mathcal{E}'$ , then  $\mathcal{E}' = \mathcal{E}$  and we are again through. To this end, suppose  $G_{K+1}$  has a weakly different equilibrium  $\mathcal{E}'$  such that  $(p_i, a_j) \in E(\mathcal{E})$  and  $(p_i, a_j) \notin E(\mathcal{E}')$  and consider the following two cases.

CASE I:  $a_j$  IS ALLOCATING RESOURCE TO  $p_i$  IN  $\mathcal{E}'$ . Consider the game played on the subtree of  $G_{K+1}$  rooted at  $p_i$  and not containing  $a_j$ . Since  $a_j$  allocates more resource on  $p_i$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ , by Lemma 1  $\omega'_i \geq \omega_i$ . On the other hand,  $a_j$  must be allocating less resource to some other good  $p_k$  in  $\mathcal{E}'$  than in  $\mathcal{E}$ , so again by Lemma 1  $\omega_k \geq \omega'_k$ . Note we also have  $\omega_i > \omega_k$  and thus conclude that  $\omega'_i > \omega'_k$ ; since  $a_j$  allocates non-zero resource to  $p_i$  in  $\mathcal{E}'$ , she is not acting optimally and this gives a contradiction to the fact that  $\mathcal{E}'$  is an equilibrium.

CASE II:  $a_j$  IS NOT ALLOCATING RESOURCE TO  $p_i$  IN  $\mathcal{E}'$ . Since the subtree rooted at  $p_i$  and not containing  $a_j$  is of size at most  $N - 1$ , by induction we have  $\omega_i = \omega'_i$ . Since  $a_j$  is allocating the same total amount of resource to  $\mathcal{N}(a_j) \setminus p_i$ , there exists  $p_k$  on which  $a_j$  is allocating nonzero resource in  $\mathcal{E}$  and not allocating strictly more resource in  $\mathcal{E}'$  than in  $\mathcal{E}$ ; by Lemma 1 this implies  $\omega'_k \leq \omega_k$ . Note we also have  $\omega_i > \omega_k$  because  $(p_i, a_j) \in E(\mathcal{E})$ , and thus we have  $\omega'_i > \omega'_k$ . Consider the following two cases: CASE 1) If  $a_j$  allocates nonzero resource to  $p_k$  in  $\mathcal{E}'$  then  $\omega'_i = \omega'_k$  because  $(p_i, a_j) \notin E(\mathcal{E}')$ ; but this is a contradiction. CASE 2) If  $a_j$  allocates zero resource to  $p_k$  then there exists good  $p_l \in \mathcal{N}(a_j) \setminus \{p_i, p_k\}$  on which  $a_j$  is allocating strictly more resource in  $\mathcal{E}'$  than in  $\mathcal{E}$ . The fact that  $(p_i, a_j) \notin E(\mathcal{E}')$  implies  $\omega'_i = \omega'_l$ , so we have  $\omega'_l > \omega'_k$ ; but this is a contradiction to the fact that  $\mathcal{E}'$  is an equilibrium.

Now we conclude that  $E(\mathcal{E}) = E(\mathcal{E}')$  and this completes the proof. ■

## 4 Nash Dynamics

Pick any utility function that is increasing, concave and differentiable, say  $\mathcal{U}(x) = \sqrt{x}$ , and define potential function  $\Psi(\omega_1, \dots, \omega_n) = \sum_{i=1}^n \sqrt{\omega_i}$ . It is clear that for any agent  $a_j$ , whenever  $a_j$  updates her allocation such that increases her total utility, the potential increases as well. This proves the following theorem.



**Theorem 7** *NCGG is a potential game.*

Therefore, better/best response Nash dynamics always converge. However it is not clear how fast the convergence is as the increment in  $a_j$ 's total utility can be either larger or smaller than the increment of the potential, depending both on  $\mathcal{U}_j(\cdot)$  and the amount of resources already allocated to  $a_j$ 's neighboring goods. In the rest of the section, we present a particular Nash dynamics where we can show fast convergence to an  $\epsilon$ -approximate Nash equilibrium. We only give details for the best response Nash dynamics (Algorithm 1), and it is easy to see the same convergence result holds for the corresponding better response Nash dynamics as well. To this end we consider  $K$ -discretized version of the game, where each agent has access to a total of  $K$  identical *atomic* resources, each of volume  $1/K$ . We start by giving the following two lemmas.

**Lemma 2** *A solution to the  $K$ -discretized CGP is optimal iff the following two conditions are satisfied: 1) the agent has allocated all of its  $K$  atomic units of resource; 2) for any two goods  $p_i, p_j \in P$ ,  $\omega_i - \omega_j > 1/K$  (where  $\omega_i = \alpha_i + x_i$  and  $\omega_j = \alpha_j + x_j$ ) implies  $x_i = 0$ .*

**Proof:** First we prove the ‘only if’ direction. It is obvious that an optimal solution must have allocated all of its  $K$  atomic units of resource because the utility function is increasing, so we focus on the proof of the second condition. Suppose otherwise we have  $p_i, p_j \in P$  with  $\omega_i - \omega_j > 1/k$ , where  $\omega_i = \alpha_i + x_i$ ,  $\omega_j = \alpha_j + x_j$  and  $x_i > 0$ . Construct another solution by moving one atomic unit of resource from good  $p_i$  to  $p_j$  gives a new solution of total utility strictly higher because the utility function is increasing and concave. Therefore we have a contradiction.

Next we prove the ‘if’ direction of the lemma. Suppose the solution  $x$  is not optimal. Let  $\omega_k$  and  $\omega'_k$  (where  $p_k \in P$ ) denote the total resource induced by this ‘suboptimal’ solution and a true optimal solution  $x'$ , respectively. Since an optimal solution must have allocated all of its  $K$  units of atomic resource among the goods, it must be true that there exist  $p_i, p_j \in P$  such that  $\omega_i - \omega'_i \geq 1/K$  and  $\omega'_j - \omega_j \geq 1/K$ , and if both inequality holds in equality, then  $\omega'_i \neq \omega_j$  (because otherwise  $x$  and  $x'$  are essentially the same, which means  $x$  is already optimal). Note  $\omega'_j - \omega_j \geq 1/K$  implies that good  $j'$  has resource allocated to it in the optimal solution (i.e.  $x'_j \geq 1/K$ ), so by the ‘only if’ part of proof above, we must have  $\omega'_i \geq \omega'_j - 1/K$ . Now we show that  $\omega_i - \omega_j > 1/K$  by considering the following two cases:

CASE I:  $(\omega_i - \omega'_i) + (\omega'_j - \omega_j) > 2/K$ . In this case, it is easily checked that  $\omega_i - \omega_j > 1/K$ .

CASE II:  $\omega_i - \omega'_i = 1/K$  AND  $\omega'_j - \omega_j = 1/K$ . As discussed above, we must not have  $\omega'_i = \omega_j$ . In fact, we must have  $\omega'_i > \omega_j$  because otherwise we will have  $\omega'_j = \omega_j + 1/K > \omega'_i + 1/K$ , which is a contradiction to optimality because  $x'_j \geq 1/K$ . Therefore, again we have reached the conclusion that  $\omega_i - \omega_j > 1/K$ .

Now note  $\omega_i - \omega'_i \geq 1/K$  implies  $x_i \geq 1/K$ , but this is a contradiction to  $\omega_i - \omega_j > 1/K$ , which by assumption implies  $x_i = 0$ . Therefore,  $x$  must itself be an optimal solution. ■

**Lemma 3** *For any  $\epsilon > 0$ , an optimal solution to the  $K$ -discretized common goods problem, where  $K = 1/\mathcal{U}^{-1}(\epsilon/n)$ , is an  $\epsilon$ -approximation to the optimal solution in the continuous common goods problem.*

**Proof:** Denote by  $OPT$  and  $OPT_K$  the optimal utility attained by an optimal solution in the continuous version and the  $K$ -discretized version, respectively; denote by  $\mathcal{W}^*$  and  $\mathcal{W}_K^*$  the set of goods to which non-zero resource is allocated in the two optimal solutions, respectively. By Lemma 2, any two goods in  $\mathcal{W}_K^*$  must have their total resources allocated differ by at most  $1/K$ , i.e.  $\omega_{max} - \omega_{min} \leq 1/K$ , where  $\omega_{min} = \min\{\omega_i \mid p_i \in \mathcal{W}_K^*\}$  and  $\omega_{max} = \max\{\omega_i \mid p_i \in \mathcal{W}_K^*\}$ . Since the agent has access to  $n$  goods, it must be the case that  $\omega_{min} \geq 1/n - 1/K$  because otherwise  $\omega_{max} < 1/n$ . Now consider the set  $\mathcal{W} = \mathcal{W}_K^* \cup \{p_i \notin \mathcal{W}_K^* \mid \alpha_i \leq \omega_{max}\}$  of goods whose total

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**Algorithm 1**  $K$ -discretized Best Response Nash Dynamics

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1: // INPUT:  $G$ ,  $\alpha \geq 0$ ,  $\epsilon > 0$ , and schedule  $\sigma$ 
2: // OUTPUT: An  $\epsilon$ -approximate Nash Equilibrium
3: Start by setting  $K = \max_{j \in [m]} \mathcal{U}_j^{-1}(\epsilon/n)$ 
4: // Set an arbitrary initial state  $s = (s_1, s_2, \dots, s_m)$ 
5: for  $j = 1$  to  $m$  do
6:    $a_j$  discretizes his one unit of resource into  $2K$  atomic units, each of volume  $1/2K$ ; arbitrarily
   assigns them to her adjacent goods, resulting in  $s_j$ 
7: end for
8: // Sort in non-increasing order of total resource allocated
9: Arrange goods in the order  $p_{\pi(1)}, p_{\pi(2)}, \dots, p_{\pi(n)}$  s.t.  $\omega_{\pi(i)} \geq \omega_{\pi(j)}$  if  $1 \leq i < j \leq n$ 
10: // Best response Nash Dynamics
11: for  $t = 1$  to  $T$  do
12:   Let  $a_{\sigma(t)}$  be the agent active in round  $t$ ;
13:   while  $\exists 1 \leq i < j \leq n$  s.t.  $\omega_{\pi(i)} - \omega_{\pi(j)} \geq 1/K$  and  $x_{\sigma(t)\pi(i)} > 0$  do
14:      $x_{\sigma(t)\pi(i)} = x_{\sigma(t)\pi(i)} - 1/2K$ ;  $x_{\sigma(t)\pi(j)} = x_{\sigma(t)\pi(j)} + 1/2K$ 
15:     If necessary, re-define  $\pi$  to maintain total resource allocated in non-increasing order.
16:   end while
17: end for
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resource is at most  $\omega_{max}$ , it is clear that: 1)  $\mathcal{W}^*$ , the optimal solution to the continuous version of the problem, forms a subset of  $\mathcal{W}$ ; 2)  $\max\{\omega_i \mid i \in \mathcal{W}^*\} \leq \omega_{max}$ .

Now suppose we have access to an additional of  $|\mathcal{W}|$  atomic units of resource, each of volume  $1/K$ , construct a new allocation by doing the following: Start with an allocation same as  $\mathcal{W}_K^*$ , then assign one atomic unit of resource to each good in  $\mathcal{W} \supseteq \mathcal{W}_K^*$ . It is clear from the above discussion that for any good  $p_i \in \mathcal{W}$ , its total resource under the new allocation is at least that of the total resource allocated under  $\mathcal{W}^*$ , which means the utility that we obtain under the new allocation,  $OPT''$ , is at least  $OPT$ . Therefore  $OPT - OPT_K \leq OPT'' - OPT_K \leq n\mathcal{U}(1/K)$ ; so to upper bound  $OPT - OPT_K$  by  $\epsilon$ , it is sufficient to set  $K = 1/\mathcal{U}^{-1}(\epsilon/n)$ . ■

Note for most reasonable choices of  $\mathcal{U}$  (e.g.  $\mathcal{U}(x) = x^p$  where  $p \in (0, 1)$ ),  $K$  is polynomial in  $n$ . We have the following theorem.

**Theorem 8** For any  $\epsilon > 0$ , Algorithm 1 converges to an  $\epsilon$ -approximate Nash equilibrium in  $O(Kmn)$  time, where  $K = \max_{j \in [m]} \mathcal{U}_j^{-1}(\epsilon/n)$ , for any updating schedule  $\sigma$ .<sup>2</sup>

**Proof:** First note according to the characterization of Lemma 2, the response of each agent  $a_{\sigma(t)}$  in Algorithm 1 is a  $K$ -discretized best response. The rest of this proof is to define a potential function<sup>3</sup> whose range are positive integers that span an interval no greater than  $Kmn$ , and to show each time an agent updates his allocation with a best response, the value of this potential function strictly decreases.

For simplicity of exposition, we write  $p_i$  in place of  $p_{\pi(i)}$  in the rest of the proof. Let  $p_1, p_2, \dots, p_n$  be the  $n$  goods arranged in non-increasing order of total resource allocated, that is,  $\omega_1 \geq \omega_2 \geq \dots \geq \omega_n$ . Define potential function  $\Phi(\omega_1, \omega_2, \dots, \omega_n) = \sum_{i=1}^n (n-i) \cdot \omega_i$ . Apparently,  $\Phi(\cdot)$  is a positive integer valued function and the difference between the greatest and smallest function value is upper bounded by  $Kmn$ . We are done if we can show that for any node  $a_{\sigma(t)}$ , the computation that  $a_{\sigma(t)}$  does on line 13-17 of Algorithm 1 results in a strict decrease in the potential.

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<sup>2</sup> $\sigma$  is assumed to at any time only pick an agent whose state is not already a best-response.

<sup>3</sup>This potential function is different from the one given in the proof of Theorem 7; this new potential function is convenient in upper bounding the convergence time.

On line 14-15 of Algorithm 1, an atomic unit of resource of volume  $1/2K$  is moved from good  $p_i$  to  $p_j$ . In doing so, the goods may no longer be sorted in non-increasing order of total resource, and in this case we restore it on line 16 of Algorithm 1, which without loss of generality can be thought of as moving  $p_i$  to the right for some  $\mu \geq 0$  positions (with  $\mu$  being the minimum necessary), and moving  $p_j$  to the left in the ordering for some  $\nu \geq 0$  positions (again with  $\nu$  being the minimum necessary). This results in the new ordering of the goods:

$$p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{i+\mu}, p_i, \dots, p_j, p_{j-\nu}, \dots, p_{j-1}, p_{j+1}, \dots, p_n$$

Note  $p_i$  still precedes  $p_j$  (i.e.  $i + \mu < j - \nu$ ) in this ordering because prior to line 14-15 of Algorithm 1,  $\omega_i - \omega_j \geq 1/K$ , therefore, the total resource of  $p_i$  is still at least that of  $p_j$  after a  $1/2K$  amount of resource has been moved from  $p_i$  to  $p_j$ . With this observation, we can analyze the change in potential by looking at the changes of potential on  $\{p_i, \dots, p_{i+\mu}\}$  and  $\{p_{j-\nu}, \dots, p_j\}$  separately, and ignore the rest of the goods, whose contribution to potential remain unchanged. Clearly, the contribution to potential from  $\{p_i, \dots, p_{i+\mu}\}$  decreases, and by an amount of  $\Delta\Phi_\downarrow = (\omega_i - \omega_{i+1}) \cdot (n - i) + (\omega_{i+1} - \omega_{i+2}) \cdot (n - i - 1) + \dots + (\omega_{i+\mu} - \omega_{i+\mu+1}) \cdot (n - i - \mu) \geq (n - i - \mu)/2K$ . Similarly, the contribution to potential from  $\{p_{j-\nu}, \dots, p_j\}$  increases by  $\Delta\Phi_\uparrow = (\omega_{j-1} - \omega_j) \cdot (n - j) + (\omega_{j-2} - \omega_{j-1}) \cdot (n - j + 1) + \dots + (\omega_j + 1/2K - \omega_{j-\nu}) \cdot (n - j + \nu) \leq (n - j + \nu)/2K$ . Since  $i + \mu < j - \nu$ , we have  $\Delta\Phi_\downarrow > \Delta\Phi_\uparrow$ , which means the potential decrease by at least 1. Therefore, in at most  $Kmn$  steps Algorithm 1 converges to a Nash equilibrium in the  $K$ -discretized game. By Lemma 3, this constitutes an  $\epsilon$ -approximate Nash equilibrium to the original game. ■

## 5 Price of Anarchy of the Game

We show in this section the *price of anarchy* of NCGG is unbounded, and it is for a reason that echoes the well-known phenomenon called *tragedy of the commons* [11].

**Theorem 9** *The price of anarchy of NCGG is  $\Omega(n^{1-\epsilon})$ , for any  $\epsilon > 0$ .*

**Proof:** Consider the bipartite graph  $G = (P, A, E)$  where  $P = \{p_c, p_1, \dots, p_n\}$ ,  $A = \{a_1, \dots, a_n\}$  and  $E = \{(p_j, a_j), (p_c, a_j) \mid j \in [n]\}$  so that all agents share the ‘common’ good  $p_c$  and each agent  $a_j$  has a ‘private’ good  $p_j$  to himself. Assume each agent  $a_j$  has the same utility function  $\mathcal{U}$ ,  $\alpha_i = 0 \forall a_i \in \{p_1, \dots, p_n\}$  and  $\alpha_c = 1$ .

It is clear that it is a Nash equilibrium for every agent  $a_j$  to allocate her entire unit of resource to her private good  $p_j$ . And in this case the social welfare is  $2n \cdot \mathcal{U}(1)$ . On the other hand, if every agent devotes her entire unit of resource to the common good, then the social welfare is  $n \cdot \mathcal{U}(n+1)$ . Therefore the price of anarchy of this particular example is at least  $\frac{\mathcal{U}(n+1)}{2\mathcal{U}(1)} = O(\mathcal{U}(n+1))$ . Since  $\mathcal{U}(\cdot)$  is concave, we can set  $\mathcal{U}(x) = x^{1-\epsilon}$ ; therefore the theorem follows. ■

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## A Proof of Theorem 1

**Proof:** Let  $\lambda_i$  ( $i = 1, 2, \dots, n$ ) be the *Lagrange multiplier* associated with the inequality constraint  $x_i \geq 0$  and  $\nu$  the *Lagrange multiplier* associated with the equality constraint  $\sum_{i=1}^n x_i = 1$ . Since the above program is convex, the following KKT optimality conditions,

$$\lambda_i^* \geq 0, x_i^* \geq 0 \quad (i \in [n]) \quad (2a)$$

$$\sum_{i=1}^n x_i^* = 1 \quad (2b)$$

$$\lambda_i^* x_i^* = 0 \quad (i \in [n]) \quad (2c)$$

$$-\frac{d}{dx_i} \mathcal{U}(\alpha_i + x_i^*) - \lambda_i^* + \nu^* = 0 \quad (i \in [n]) \quad (2d)$$

are sufficient and necessary for  $x^*$  to be the optimal solution to the (primal) convex program (1) and  $(\lambda^*, \nu^*)$  the optimal solution to the associated dual program.

Let  $\mathcal{V}$  be the inverse function of  $\frac{d}{dx} \mathcal{U}$ . Note equation (2c) and (2d) implies  $(-\frac{d}{dx_i} \mathcal{U}(\alpha_i + x_i^*) + \nu^*) x_i^* = 0$ ; equation (2a) and (2d) implies  $-\frac{d}{dx_i} \mathcal{U}(\alpha_i + x_i^*) + \nu^* \geq 0$ , which combining with the fact that  $\mathcal{U}(\cdot)$  is convex implies  $\mathcal{V}(\nu^*) \leq \alpha_i + x_i^*$ . If  $\alpha_i < \mathcal{V}(\nu^*)$ , then  $x_i^* > 0$  and thus  $-\frac{d}{dx_i} \mathcal{U}(\alpha_i + x_i^*) + \nu^* = 0$ , i.e.  $x_i^* = \mathcal{V}(\nu^*) - \alpha_i$ . On the other hand, if  $\alpha_i \geq \mathcal{V}(\nu^*)$ , then we must have  $x_i^* = 0$ . To see why this is true, suppose otherwise  $x_i^* > 0$ ; this leads to  $x_i^* = \mathcal{V}(\nu^*) - \alpha_i \leq 0$ , which is a contradiction. We summarize the optimal solution  $x^*$  as follows

$$x_i^* = \begin{cases} \mathcal{V}(\nu^*) - \alpha_i & \alpha_i < \mathcal{V}(\nu^*) \\ 0 & \alpha_i \geq \mathcal{V}(\nu^*) \end{cases}$$

where  $\nu^*$  is a solution to  $\sum_{i=1}^n \max\{0, \mathcal{V}(\nu^*) - \alpha_i\} = 1$ .

It is easy to see that  $\sum_{i=1}^n \max\{0, \mathcal{V}(\nu^*) - \alpha_i\} = 1$  admits a unique solution if we treat  $\mathcal{V}(\nu^*)$  as the variable, i.e. different utility functions only leads to different solutions of the Lagrange multiplier  $\nu^*$  but  $\mathcal{V}(\nu^*)$  remains invariant. Therefore the optimal solution  $x^*$  is unique not only of a particular choice of  $\mathcal{U}(\cdot)$ , but across all utility functions that are increasing, concave and differentiable.  $\blacksquare$